TUTORIAL NOTES FOR MATH4220

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1. THE MAXIMUM PRINCIPLES (HEAT EQUATION)

Recall the maximum principles.

1.1. Notations.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, we set $\Omega_T := (0,T] \times \Omega = (t,x) : 0 < t \le T, x \in \Omega$, we denote by $\partial_p \Omega_T$ the parabolic boundary of Ω_T which is defined as $\partial_p \Omega_T := \{t = 0\} \times \bar{\Omega} \cup (0,T] \times \partial\Omega$. As usual, we denote B_r to be the ball with radius r in $\mathbb{R}^n$, we set $Q_r := (-r^2, 0] \times B_r$. Moreover, we denote by $C^{a,b}(\Omega_T)$ the collection of functions in Ω_T which are C^a in t and C^b in x.

1.2. Theorems.

Theorem 1 (Weak maximum principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and T be a positive constant. Let $u \in C^{1,2}(\Omega_T) \cap C(\bar{\Omega}_T)$ with $u_t - \Delta u \le 0$ (≥ 0) in Ω. Then,*

$$\sup_{\Omega_T} u = \sup_{\partial_p \Omega_T} u \ (\inf_{\Omega_T} u = \inf_{\partial_p \Omega_T} u).$$

Consequently, for u satisfies $u_t - \Delta u = 0$,

$$\inf_{\partial_p \Omega_T} u \le u(x) \le \sup_{\partial_p \Omega_T} u, \quad \forall x \in \Omega.$$

Theorem 2 (Strong maximum principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and T be a positive constant. Let $u \in C^{1,2}(\Omega_T) \cap C(\bar{\Omega}_T)$ with $u_t - \Delta u \le 0$ (≥ 0) in Ω, and suppose there exists a point $(\tau, y) \in \Omega_T$ for which $u(\tau, y) = \sup_{\Omega_T} u$ $(\inf_{\Omega} u)$. Then $u \equiv u(\tau, y)$ is constant in $(0, \tau) \times \Omega$. Consequently the solution to heat equation cannot assume an interior maximum or minimum value unless it is constant.*

1.3. Applications.

Let us show some results given by the mean value theorems and the maximum principles.

Example 3 (Interior gradient estimates)**.** Suppose $u \in C^{1,2}(Q_1) \cap C(\bar{Q}_1)$ satisfies

$$u_t - \Delta u = 0 \quad \text{in } Q_1.$$

Then there holds

$$\sup_{Q_{\frac{1}{2}}} |Du| \le c \sup_{\partial_p Q_1} |u|,$$

where $c = c(n)$ is a positive constant. In particular for any $\alpha \in [0,1]$ there holds

$$|u(t,x) - u(t,y)| \le c|x-y|^\alpha \sup_{\partial_p Q_1} |u|, \quad \forall t > 0, x, y \in B_{\frac{1}{2}},$$

where $c = c(n, \alpha)$ is a positive constant.

Proof. Direct computation shows that

$$(\partial_t - \Delta)(|Du|^2) = -2\sum_{i,j=1}^n (D_{ij}u)^2 + 2\sum_{i=1}^n D_iuD_i(\partial_t u - \Delta u) = -2\sum_{i,j=1}^n (D_{ij}u)^2,$$

moreover,

$$(\partial_t - \Delta)(\varphi|Du|^2) = (\partial_t\varphi - \Delta\varphi)|Du|^2 - 4\sum_{i,j=1}^n D_i\varphi D_juD_{ij}u - 2\varphi\sum_{i,j=1}^n (D_{ij}u)^2, \quad \forall \varphi \in C_0^1(B_1).$$

By taking $\varphi = \eta^2$ for some $\eta \in C_0^1(B_1)$ with $\eta \equiv 1$ in $B_{\frac{1}{2}}$, we obtain by the Hölder's inequality,

$$\begin{aligned}(\partial_t - \Delta)(\eta^2|Du|^2) =&(2\eta\partial_t\eta - 2\eta\Delta\eta - 2|D\eta|^2)|Du|^2 \\ &- 8\eta\sum_{i,j=1}^n D_i\eta D_juD_{ij}u - 2\eta^2\sum_{i,j=1}^n (D_{ij}u)^2 \\ \le&(2\eta\partial_t\eta - 2\eta\Delta\eta + 6|D\eta|^2)|Du|^2 \le C|Du|^2,\end{aligned}$$

where C is a positive constant depending only on η. Moreover, since

$$(\partial_t - \Delta)(u^2) = -2|Du|^2 + 2u(\partial_t u - \Delta u) = -2|Du|^2,$$

by choosing α large enough we get

$$\Delta(\eta^2|Du|^2 + \alpha u^2) \le 0,$$

then by the maximum principle, we have

$$\sup_{Q_1}(\eta^2|Du|^2 + \alpha u^2) \le \sup_{\partial_p Q_1}(\eta^2|Du|^2 + \alpha u^2),$$

which implies

$$\sup_{Q_{\frac{1}{2}}}|Du| \le c\sup_{\partial_p Q_1}|u|,$$

where $c = c(n)$ is a positive constant. Therefore we have

$$|u(t,x) - u(t,y)| \le c|x-y|^\alpha \sup_{\partial_p Q_1}|u|, \quad \forall t > 0, x, y \in B_{\frac{1}{2}}.$$

□

Example 4 (Li-Yau estimate)**.** Suppose u is a non-negative function satisfies

$$\partial_t u - \Delta u = 0 \quad \text{in } (0,T] \times \mathbb{R}^n.$$

Then there holds

$$\frac{|Du|^2}{u^2} - \frac{\partial_t u}{u} \le \frac{n}{2t},$$

In particular, for arbitrary (t_1, x_1), $(t_2, x_2) \in (0,T] \times \mathbb{R}^n$ with $t_2 > t_1 > 0$, there holds

$$\frac{u(t_1,x_1)}{u(t_2,x_2)} \le \left(\frac{t_2}{t_1}\right)^{\frac{n}{2}} e^{\frac{|x_2-x_1|^2}{4(t_2-t_1)}}.$$

Proof. We divide the proof into several steps.

Firstly, we derive some equations involving $v = \log u$. For v, we have

$$\partial_t v - \Delta v = |Dv|^2,$$

moreover, denote $w = |Dv|^2$, then

$$\partial_t w - \Delta w = 2Dv \cdot Dw - 2 \sum_{i,j=1}^{n} (D_{ij}v)^2.$$

Secondly, for $\alpha \in (0,1)$, set $f = \alpha|Dv|^2 - v_t$, then

$$\partial_t f - \Delta f = 2Dv \cdot Df - 2\alpha \sum_{i,j=1}^{n} (D_{ij}v)^2.$$

Since

$$\begin{aligned}\sum_{i,j=1}^{n} (D_{ij}v)^2 \geq & \frac{1}{n}(\Delta v)^2 = \frac{1}{n}(|Dv|^2 - v_t) = \frac{1}{n}\left((1-\alpha)|Dv|^2 + f\right)^2 \\ \geq & \frac{1}{n}\left(f^2 + 2(1-\alpha)|Dv|^2 f + (1-\alpha)^2|Dv|^4\right) \\ \geq & \frac{1}{n}\left(f^2 + 2(1-\alpha)|Dv|^2 f\right),\end{aligned}$$

therefore

$$\partial_t f - \Delta f \leq 2Dv \cdot Df - \frac{2\alpha}{n}\left(f^2 + 2(1-\alpha)|Dv|^2 f\right).$$

Thirdly, for arbitrary $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi \geq 0$, let $g = t\varphi f$, we have

$$\begin{aligned} t\varphi\partial_t f &= g_t - \frac{g}{t}, \\ t\varphi Df &= Dg - \frac{D\varphi}{\varphi}g, \\ t\varphi\Delta f &= \Delta g - 2\frac{D\varphi}{\varphi}\cdot Dg + \left(2\frac{|D\varphi|^2}{\varphi^2} - \frac{\Delta\varphi}{\varphi}\right)g. \end{aligned}$$

Therefore substituting $\partial_t f$, Df, Δf by the above relations, we obtain

$$\begin{aligned} & t\varphi(\partial_t g - \Delta g) + t(D\varphi - \varphi Dv)\cdot Dg \\ \leq & g\left[\varphi - \frac{2\alpha}{n}g + t\left(2\frac{|D\varphi|^2}{\varphi} - \Delta\varphi + \frac{n}{4\alpha(1-\alpha)}\frac{|D\varphi|^2}{\varphi}\right)\right]. \end{aligned}$$

By letting $\varphi = \eta^2$ for $\eta \in C_0^\infty(\mathbb{R}^n)$ with $\eta \geq 0$, we get

$$\begin{aligned} & t\eta^2(\partial_t g - \Delta g) + t(2\eta D\eta - \eta^2 Dv)\cdot Dg \\ \leq & g\left[\eta^2 - \frac{2\alpha}{n}g + t\left(6|D\eta|^2 - 2\eta\Delta\eta + \frac{n}{\alpha(1-\alpha)}|D\eta|^2\right)\right]. \end{aligned}$$

Let η_0 be a cutoff function with $0 \leq \eta_0 \leq 1$ in B_1 and $\eta_0 \equiv 1$ in $B_{\frac{1}{2}}$, then let $\eta(x) = \eta_0(\frac{x}{R})$, we have

$$t\eta^2(\partial_t g - \Delta g) + t(2\eta D\eta - \eta^2 Dv)\cdot Dg \leq g\left(1 - \frac{2\alpha}{n}g + \frac{C_\alpha t}{R^2}\right).$$

We claim that

$$1 - \frac{2\alpha}{n}g + \frac{C_\alpha t}{R^2} \geq 0 \quad \text{in } (0,T]\times B_R. \tag{1.1}$$

Denote $h := 1 - \frac{2\alpha}{n}g + \frac{C_\alpha t}{R^2}$, suppose to the contrary, that h has a negative minimum at $(t_0, x_0) \in Q_T$, then $h(t_0,x_0) < 0$, $\partial_t h(t_0,x_0) \leq 0$, $Dh(t_0,x_0) = 0$, and

$\Delta h(t_0,x_0) \geq 0$, therefore $g(t_0,x_0) > 0$, $\partial_t g(t_0,x_0) \geq 0$, $Dg(t_0,x_0) = 0$, and $\Delta g(t_0,x_0) \leq 0$. Then on the one hand, at (t_0,x_0), we have

$$t\eta^2(\partial_t g - \Delta g) + t(2\eta D\eta - \eta^2 Dv)\cdot Dg \geq 0,$$

on the other hand, we also have

$$t\eta^2(\partial_t g - \Delta g) + t(2\eta D\eta - \eta^2 Dv)\cdot Dg \leq g\left(1 - \frac{2\alpha}{n}g + \frac{C_\alpha t}{R^2}\right) < 0,$$

which is a contradiction. Let R goes to infinity in (1.1), we have

$$\frac{|Du|^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{n}{2t}.$$

Finally, to prove the Harnack inequality, for $(t_1,x_1), (t_2,x_2) \in (0,T]\times\mathbb{R}^n$, take an arbitrary path $x(t)$ for $t\in[t_1,t_2]$ with $x(t_1)=x_1$, $x(t_2)=x_2$, then

$$\begin{aligned}\frac{dv(t,x(t))}{dt} =& v_t + Dv\cdot\frac{dx(t)}{dt}\\ \geq& |Dv|^2 + Dv\cdot\frac{dx(t)}{dt} - \frac{n}{2t}\\ \geq& -\frac{1}{4}\left|\frac{dx(t)}{dt}\right|^2 - \frac{n}{2t}.\end{aligned}$$

then

$$v(t_1,x_1) \leq v(t_2,x_2) + \frac{n}{2}\log\frac{t_2}{t_1} + \frac{1}{4}\int_{t_1}^{t_2}\left|\frac{dx(t)}{dt}\right|^2 dt.$$

Let $x(t) = at+b$, where

$$a = \frac{x_2-x_1}{t_2-t_1}, \quad b = \frac{t_2x_1 - t_1x_2}{t_2-t_1}.$$

Then

$$v(t_1,x_1) \leq v(t_2,x_2) + \frac{n}{2}\log\frac{t_2}{t_1} + \frac{1}{4}\frac{|x_2-x_1|^2}{t_2-t_1},$$

therefore

$$\frac{u(t_1,x_1)}{u(t_2,x_2)} \leq \left(\frac{t_2}{t_1}\right)^{\frac{n}{2}} e^{\frac{|x_2-x_1|^2}{4(t_2-t_1)}}.$$

□

A Supplementary Problem

For a bounded region $\Omega\subset\mathbb{R}^n$ and a positive constant T, if u satisfies

$$\begin{aligned}\partial_t u - \Delta u =& f, \quad \text{in } \Omega_T,\\ u(0,\cdot) =& u_0, \quad \text{on } \Omega,\\ u =& \varphi, \quad \text{on } (0,T)\times\partial\Omega.\end{aligned}$$

Show that

$$\sup_{\Omega_T}|u| \leq C\left(\sup_{\Omega}|u_0| + \sup_{[0,T]\times\partial\Omega}|\varphi| + \sup_{\Omega_T}|f|\right),$$

where $C = C(T,\Omega)$ is a positive constant.

For more materials, please refer to [1, 2, 3, 4].

References

[1] S. Alinhac, *Hyperbolic partial differential equations*, Universitext, Springer, Dordrecht, 2009.
[2] L. C. Evans, *Partial differential equations*, vol. 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 1998.
[3] Q. Han and F. Lin, *Elliptic partial differential equations*, vol. 1 of Courant Lecture Notes in Mathematics, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1997.
[4] W. A. Strauss, *Partial differential equations. An introduction*, John Wiley & Sons, Inc., New York, 1992.

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